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Best Approximation by Normal and Conormal Sets

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The aim of the present paper is to develop a theory of best approximation by elements of so-called normal sets and their complements—conormal sets—in the non-negative orthant $\mathbb{R}^I_+$ of a finite-dimensional coordinate space $\mathbb{R}^I$ endowed with the max-norm. A normal (respectively, conormal) set arises as the set of all solutions of a system of inequalities $f_\alpha(x) \leqslant 0$ $(\alpha \in A)$, $x \in \mathbb{R}^I_+$ (respectively, $f_\alpha(x) \geqslant 0$ $(\alpha \in A)$, $x \in \mathbb{R}^I_+$), where f_α is an increasing function and A is an arbitrary set of indices. We consider these sets as analogues (in a certain sense) of convex sets, and we use the so-called min-type functions as analogues of linear functions. We show that many results on best approximation by convex and reverse convex sets and corresponding separation theory (but not all of them) have analogues in the case under consideration. At the same time there are no convex analogues for many results related to best approximation by normal sets. © 2000 Academic Press

1. INTRODUCTION

If $X = (X, \|\cdot\|)$ is a normed linear space, G a subset of X, and $x^0 \in X$, the *distance* of x^0 to G is the number

$$\operatorname{dist}(x^0, G) := \inf_{g \in G} \|x^0 - g\|, \tag{1.1}$$

and an element $g^0 \in G$ is called an *element of best approximation* of x^0 by the set G if it is the "nearest" to x^0 among the elements of G, i.e., if

$$\|x^0 - g^0\| = \min_{g \in G} \|x^0 - g\| \tag{1.2}$$

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(we shall denote by min and max an inf, respectively a sup, which is attained); the set of all such elements g^0 will be denoted by $P_G(x^0)$. It is well known that if the dimension of X is finite and G is closed, then $P_G(x^0) \neq \emptyset$ (where $\emptyset$ denotes the empty set).

The theory of best approximation by elements of convex sets and reverse convex sets (i.e., complements of convex sets) in normed linear spaces, which has many important applications in mathematics and other sciences, is well developed (see, e.g., [9, 10] and the references therein). However, convexity or reverse convexity is sometimes a very restrictive assumption, and therefore there arises the problem of finding other classes of sets, at least in some classes of spaces, useful in various applications, for which a theory of best approximation can be developed; also, it is desirable to use, as much as possible, some analogues of the methods of the theory of best approximation by convex and reverse convex sets.

The aim of the present paper is to develop a theory of best approximation by elements of a class of non-convex sets and their complements in the cone $\mathbb{R}^n_+$ (of all elements with non-negative coordinates of the finite-dimensional space $\mathbb{R}^n$), endowed with a suitable norm, namely the so-called normal sets and their complements. We shall use, instead of $\mathbb{R}^n_+$ and $\mathbb{R}^n$, the notations $\mathbb{R}^I_+$ and $\mathbb{R}^I$, respectively, where I is a finite index set, since we want to emphasize that some of our results and proofs remain valid for bounded functions on an arbitrary index set I; however, in this paper we shall assume that I is finite. We recall that a subset G of $\mathbb{R}^I_+$ is called *normal* [2, 6, 7] if

$$g \in G, \qquad (0 \leqslant)\ x \leqslant g \Rightarrow x \in G. \tag{1.3}$$

We shall say that a subset G of $\mathbb{R}^I_+$ is *conormal* (or *reverse normal*) if $@G := \mathbb{R}^I_+ \setminus G$ is a normal set.

It is well known (and easy to check) that a set $G \subseteq \mathbb{R}^I_+$ is normal (respectively, conormal) if and only if it is the solution set of a system of inequalities $f_\alpha(x) \leqslant 0$ $(\alpha \in A)$ (respectively, $f_\alpha(x) \geqslant 0$ $(\alpha \in A)$), where $(f_\alpha)_{\alpha \in A}$ is a family of increasing functions defined on $\mathbb{R}^I_+$ and A is an arbitrary set of indices. Normal sets have found many applications, e.g., in mathematical economics, where they are usually called *sets with free disposal* (see [2]; see also [3] for applications of non-convex normal sets to the so-called von Neumann dynamics).

The most suitable norm to develop our theory of best approximation in $\mathbb{R}^I_+$ by normal and conormal sets will be the "l^∞-norm" (or "max-norm") of $\mathbb{R}^I$; i.e.,

$$\|x\|_\infty = \max_{i \in I} |x_i| \qquad (x = (x_i)_{i \in I} \in \mathbb{R}^I). \tag{1.4}$$

The main tool used in the theory of best approximation of an element x^0 in a normed linear space X by a closed convex set G is the separation of G and the ball $B(x^0, r)$ with center x^0 and radius $r := \text{dist}(x^0, G) > 0$ by continuous linear functions (or, equivalently, by closed hyperplanes or closed half-spaces); it is well known that the separability of G and the ball $B(x^0, r)$ is a consequence of the separability of G and any outside point. In this paper we shall show that there exists such a tool also for the study of best approximation of an element x^0 in $\mathbb{R}^I_+$ by a closed normal set G, namely the separation of G and the ball $B(x^0, r)$ by so-called "min-type functions" (see their definition in Section 2.1, formula (2.2)), which play now the role of the linear functions; i.e., in this problem the min-type functions (or, equivalently, the min-type hyperplanes or min-type half-spaces) are the appropriate class of "surrogate linear functions" (respectively, "surrogate hyperplanes" or "surrogate half-spaces") to be used for separation. However, although each closed normal set G and each outside point can be separated by a min-type function, the separability of G and the above ball $B(x^0, r)$ is no longer a consequence of this fact. It will turn out that there are also a number of other differences from the convex case, due to some special situations which may occur, and some results which we shall obtain for best approximation by normal sets in $\mathbb{R}^I_+$ do not admit analogues in the convex theory. For example, we shall show that in the theory of best approximation by normal sets the least element of best approximation g^0 plays an important role; also, the separability of the closed normal set G and the ball $B(x^0, r)$ by a min-type function, mentioned above, is sufficient, but no longer necessary, for g^0 to be an element of best approximation to $x^0 \in @G$, unless g^0 is a "weak Pareto point" of G (see Definition 2.2). Let us also note the difference between the methods and tools used for normal sets and conormal sets: while in the study of best approximation by normal sets we use "direct methods," the main tool being separation in $\mathbb{R}^I_+$, we study best approximation by conormal sets as the optimization problem of minimizing the function $f(y) := \|x^0 - y\|$ $(y \in \mathbb{R}^I_+)$ on a conormal set, using some tools of optimization theory and the theory of abstract convexity.

The structure of the paper is as follows. In Section 2 we present some preliminary results concerning increasing positively homogeneous functions, min-type functions, normal and conormal sets, and min-type half-spaces and hyperplanes. In Section 3 we show that there exists the least element of best approximation by a normal set and we compute it. The next two sections contain the main results of the paper: min-type separation of a closed normal set and a ball is discussed in Section 4, and characterizations of nearest points in closed normal sets are given in Section 5. Section 6 contains formulae for calculation of the distance to a min-type hyperplane, a min-type lower half-space, and a normal set. In the final Section 7 we discuss best approximation by conormal sets.

2. PRELIMINARIES

2.1. *IPH Functions and Min-Type Functions*

Let I be a finite set of indices. Consider the space $\mathbb{R}^I$ of all vectors $(x_i)_{i \in I}$. We shall use the following notations:

- x_i is the ith coordinate of a vector $x \in \mathbb{R}^I$;
- if $x, y \in \mathbb{R}^I$ then $x \geqslant y \Leftrightarrow x_i \geqslant y_i$ for all $i \in I$;
- if $x, y \in \mathbb{R}^I$ then $x \gg y \Leftrightarrow x_i > y_i$ for all $i \in I$;
- $\mathbb{R}^I_+ := \{x = (x_i)_{i \in I} \in \mathbb{R}^I \mid x_i \geqslant 0 \text{ for all } i \in I\}$;
- $\mathbb{R}^I_{++} := \{x = (x_i)_{i \in I} \in R^I \mid x_i > 0 \text{ for all } i \in I\}$;
- $\mathbf{1} = (1, ..., 1)$.

Due to their close connections with normal sets in $\mathbb{R}^I_+$ (see Section 2.2), the *increasing positive homogeneous* (IPH) *functions* defined on $\mathbb{R}^I_+$ will be an important tool in the sequel. We recall that a function p: $\mathbb{R}^I_+ \to \mathbb{R} \cup \{+\infty\}$ is called (a) *positively homogeneous* (of the first degree) if $p(\lambda x) = \lambda p(x)$ for all $x \in \mathbb{R}^I_+$ and $\lambda > 0$; (b) *increasing* if $x, y \in \mathbb{R}^I_+$ and $x \leqslant y$ imply $p(x) \leqslant p(y)$. We assume that the set of all IPH functions is equipped with the natural order relation: $p_1 \geqslant p_2$ if $p_1(x) \geqslant p_2(x)$ for all $x \in \mathbb{R}^I_+$.

In the sequel, for each $l \in \mathbb{R}^I_+$ we shall consider the set

$$I(l) := \{i \in I \mid l_i > 0\} \tag{2.1}$$

and we shall identify the vector l and the function $\langle l, \cdot \rangle$: $\mathbb{R}^I_+ \to \mathbb{R}$ defined by

$$\langle l, x \rangle := \begin{cases} \min_{i \in I(l)} l_i x_i \; (x \in \mathbb{R}^I_+) & \text{if} \quad I(l) \neq \varnothing \\ 0 \; (x \in \mathbb{R}^I_+) & \text{if} \quad I(l) = \varnothing. \end{cases} \tag{2.2}$$

The function $\langle l, \cdot \rangle$ defined by (2.2) is called a *min-type function*. Clearly every min-type function is IPH. We shall use the min-type functions $x \to \langle l, x \rangle$ of (2.2) to replace the linear functions $x \to \sum_{i=1}^n l_i x_i$ of the classical theory of convex optimization; therefore, the min-type functions $x \to \langle l, x \rangle$ might be called "surrogate linear functions." The set L of all min-type functions (2.2) with $l \in \mathbb{R}^I_+$ will play the role of the "conjugate space."

Remark 2.1. The "coupling function" $\varphi(x, l) := \langle l, x \rangle$, which is necessary for the study of duality over $\mathbb{R}^I_+$, is not symmetric; i.e., in general $\langle l, x \rangle \neq \langle x, l \rangle$. On the other hand, it is well known (see e.g. [6]) that the symmetric coupling function

$$\psi(x, l) := \min_{i \in I} l_i x_i \qquad (x \in \mathbb{R}^I_{++}) \tag{2.3}$$

is suitable for the study of duality over $\mathbb{R}^I_{++}$. However, we shall not study best approximation in the framework of $\mathbb{R}^I_{++}$, e.g., because the least element of best approximation, which will play a crucial role in the sequel, need not exist in $\mathbb{R}^I_{++}$ (take, for example, $I=\{1, 2\}$, $G=$ the unit square in R^2_{++}, and $x^0=(2, 1)$).

We recall that the *L-subdifferential* $\partial_L p(y)$ *of an* IPH *function* p *at a point* y, where L is the set of all min-type functions, is defined as the set

$$\partial_L p(y) := \{l \in \mathbb{R}^I_+ \mid \langle l, z\rangle \leqslant p(z)\ (z \in \mathbb{R}^I_+), \langle l, y\rangle = p(y)\}. \tag{2.4}$$

For any $x \in \mathbb{R}^I_+$ and $a \in \mathbb{R}_+$, we shall denote by $\frac{a}{x}$ the element of $\mathbb{R}^I_+$ defined by

$$\left(\frac{a}{x}\right)_i = \begin{cases} \dfrac{a}{x_i} & \text{if} \quad i \in I(x) \\ 0 & \text{if} \quad i \notin I(x). \end{cases} \tag{2.5}$$

In the remainder of this paper we shall use the following simplified version of [4, Proposition 5.1]:

PROPOSITION 2.1. *Let* p *be an* IPH *function defined on* $\mathbb{R}^I_+$ *and let* $y \in \mathbb{R}^I_+$ *such that* $0 < p(y) < +\infty$. *Then*

$$\partial_L p(y) = \left\{ l \in \mathbb{R}^I_+ \mid \langle l, y\rangle \geqslant p(y),\ p\left(\frac{1}{l}\right) = 1 \right\}. \tag{2.6}$$

Hence, in particular,

$$\frac{p(y)}{y} \in \partial_L p(y). \tag{2.7}$$

Proof. Let us denote the right hand side of (2.6) by A.
Let $l \in \partial_L p(y)$. Then, clearly, $\langle l, y\rangle \geqslant p(y)$. Define $y_l \in \mathbb{R}^I_+$ by

$$(y_l)_i := \begin{cases} y_i & \text{if} \quad i \in I(l) \\ 0 & \text{if} \quad i \notin I(l). \end{cases} \tag{2.8}$$

Since $\langle l, y\rangle \geqslant p(y)$, we have $y_i \geqslant (\frac{1}{l})_i\, p(y)$ for all $i \in I(l)$, whence also for all i, and hence, by (2.8), $y \geqslant y_l \geqslant \frac{p(y)}{l}$. Therefore, since p is IPH, we obtain

$$p(y) \geqslant p(y_l) \geqslant p\left(\frac{p(y)}{l}\right) = p(y)\, p\left(\frac{1}{l}\right),$$

whence $p(\frac{1}{l}) \leqslant 1$. On the other hand, $p(\frac{1}{l}) \geqslant \langle l, \frac{1}{l} \rangle = 1$, so $p(\frac{1}{l}) = 1$. Thus, $l \in A$.

Conversely, assume now that $l \in A$ and assume, a contrario, that there exists $z \in \mathbb{R}^I_+$ such that $\langle l, z \rangle > p(z)$. Choose $\varepsilon > 0$ such that $\langle l, z \rangle > p(z) + \varepsilon$. Then $z_i \geqslant (\frac{1}{l})_i (p(z) + \varepsilon)$ for all $i \in I(l)$, whence also for all $i \in I$, so $z \geqslant \frac{1}{l}(p(z) + \varepsilon)$. Consequently, since p is IHP and $p(\frac{1}{l}) = 1$, it follows that

$$p(z) \geqslant p\left(\frac{1}{l}(p(z) + \varepsilon)\right) = p\left(\frac{1}{l}\right)(p(z) + \varepsilon) = p(z) + \varepsilon,$$

which is impossible. This proves that $\langle l, z \rangle \leqslant p(z)$ for all $z \in \mathbb{R}^I_+$. Hence, in particular, since $l \in A$, we obtain $\langle l, y \rangle = p(y)$. Thus, $l \in \partial_L p(y)$.

Finally, let $l := \frac{p(y)}{y}$. Then $\langle l, y \rangle = \langle \frac{p(y)}{y}, y \rangle \geqslant p(y)$ and $p(\frac{1}{l}) = p(\frac{y}{p(y)}) = 1$, whence, by (2.6), $l \in \partial_L p(y)$. ∎

2.2. *Normal and Conormal Sets; Min-Type Half-Spaces and Hyperplanes*

Due to the remarks made after formula (2.2), it is natural to introduce the following terminology.

Definition 2.1. For each $l \in \mathbb{R}^I_+ \setminus \{0\}$ and $a \in \mathbb{R}_+$, the sets

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \leqslant a\}, \qquad \{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle < a\} \tag{2.9}$$

will be called *lower* (*min-type*) *half-spaces*, the sets

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \geqslant a\}, \qquad \{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle > a\} \tag{2.10}$$

will be called *upper* (*min-type*) *half-spaces*, and the set

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle = a\} \tag{2.11}$$

will be called a (*min-type*) *hyperplane*.

Let $G \subseteq \mathbb{R}^I_+$. We can consider G as a subset of the topological space $\mathbb{R}^I$ and as a subset of the topological space $\mathbb{R}^I_+$. Correspondingly, we can consider the interior, closure, and boundary of the set G with respect to $\mathbb{R}^I$ and with respect to $\mathbb{R}^I_+$. We will use the notations int G, cl G, and bd G if G is considered as a subset of $\mathbb{R}^I$ and the notations $\text{int}_0\, G$, $\text{cl}_0\, G$, and $\text{bd}_0\, G$ if G is considered as a subset of $\mathbb{R}^I_+$. Hereafter, unless otherwise stated, by "closed" (or "open") we shall mean closed (respectively, open) in the topological space $\mathbb{R}^I$.

Lemma 2.1. *For each $l \in \mathbb{R}^I_+ \setminus \{0\}$ the min-type half-spaces $\{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \leqslant 1\}$ and $\{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \geqslant 1\}$ and the min-type hyperplane $\{x \in$*

$\mathbb{R}^I_+ \mid \langle l, x\rangle = 1\}$ *are closed in the topological space* $\mathbb{R}^I_+$, *the min-type half-space* $\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle < 1\}$ *and* $\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle > 1\}$ *are open in the topological space* $\mathbb{R}^I_+$, *and we have*

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \leqslant 1\} = \mathrm{cl}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle < 1\}, \tag{2.12}$$

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \geqslant 1\} = \mathrm{cl}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle > 1\}, \tag{2.13}$$

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle < 1\} = \mathrm{int}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \leqslant 1\}, \tag{2.14}$$

$$\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle > 1\} = \mathrm{int}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \geqslant 1\}, \tag{2.15}$$

$$\begin{aligned}\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle = 1\} &= \mathrm{bd}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \leqslant 1\} \\ &= \mathrm{bd}_0\{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \geqslant 1\}.\end{aligned} \tag{2.16}$$

Proof. Clearly, it is enough to prove (2.12) and (2.13). The inclusion $\supseteq$ in (2.12) is obvious. Conversely, let $x^0 \in \mathbb{R}^I_+$, $\langle l, x^0\rangle \leqslant 1$. If $\langle l, x^0\rangle < 1$, we are done. Assume now that $\langle l, x^0\rangle = 1$, so there exists an index $j \in I(l)$ such that $l_j x^0_j = 1$, whence $j \in I(l) \cap I(x^0)$. Define $x^k \in \mathbb{R}^I_+$ $(k = 1, 2, ...)$ by

$$(x^k)_i := \begin{cases} \left(1 - \dfrac{1}{k}\right) x^0_i & \text{if} \quad i \in I(x^0) \\ 0 & \text{if} \quad i \notin I(x^0). \end{cases} \tag{2.17}$$

Then $\langle l, x^k\rangle \leqslant l_j (x^k)_j = l_j (1 - \frac{1}{k})\, x^0_j < l_j x^0_j = 1$ $(k = 1, 2, ...)$ and $x^k \to x^0$, so $x^0 \in \mathrm{cl}_0\{x \in R^I_+ \mid \langle l, x\rangle < 1\}$.

The proof of (2.13) is similar, with the only difference that if $x^0 \in \mathbb{R}^I_+$, $\langle l, x^0\rangle \geqslant 1$, then, defining $x^k \in \mathbb{R}^I_+$ $(k = 1, 2, ...)$ by

$$(x^k)_i := \begin{cases} \left(1 + \dfrac{1}{k}\right) x^0_i & \text{if} \quad i \in I(x^0) \\ 0 & \text{if} \quad i \notin I(x^0), \end{cases} \tag{2.18}$$

we have $\langle l, x^k\rangle > \langle l, x^0\rangle \geqslant 1$ $(k = 1, 2, ...)$ and $x^k \to x^0$. ∎

We have the following characterization of normal sets with the aid of min-type functions (or, equivalently, of open lower min-type half-spaces):

PROPOSITION 2.2. *For a subset* G *of* $\mathbb{R}^I_+$ *the following conditions are equivalent*:

1°. *G is normal.*

2°. *For each* $x \in @G$ *there exists* $l \in \mathbb{R}^I_+$ *such that*

$$\langle l, g\rangle < 1 = \langle l, x\rangle \qquad (g \in G). \tag{2.19}$$

Proof. If 1° holds and $x \in @G$, then

$$\left\langle \frac{1}{x}, g \right\rangle < 1 = \left\langle \frac{1}{x}, x \right\rangle \qquad (g \in G); \tag{2.20}$$

indeed, if (2.20) does not hold, i.e., if there exists $g \in G$ such that $\langle \frac{1}{x}, g \rangle \geqslant 1$, then $g \geqslant x \geqslant 0$, whence, since G is a normal set, $x \in G$, in contradiction to our assumption on x. Thus, for $l := \frac{1}{x}$ we have (2.19).

Conversely, assume now that 2° holds and G is not a normal set, so there exist $g \in G$ and $x \in @G$, with $x \leqslant g$. Then for any l as 2° we obtain $\langle l, x \rangle \leqslant \langle l, g \rangle < 1 = \langle l, x \rangle$, which is impossible. ∎

Remark 2.2. (a) Proposition 2.2 shows that a subset G of $\mathbb{R}^I_+$ is normal if and only if it is "evenly normal," i.e., an intersection of open min-type lower half-spaces; in the language of abstract convex analysis (see, e.g., [8]), this means that G is "convex with respect to the family of all open min-type lower half-spaces." This result is in striking contrast to the situation for the usual convex and evenly convex sets G in a locally convex space.

(b) For open (not necessarily normal) sets we have the following stronger result: *For any open set G in the topological space $\mathbb{R}^I_+$ and any $l \in \mathbb{R}^I_+ \setminus \{0\}$ we have*

$$\langle l, g \rangle < \sup_{g' \in G} \langle l, g' \rangle \qquad (g \in G). \tag{2.21}$$

Indeed, let $l \in \mathbb{R}^I_+ \setminus \{0\}$, $g \in G$. Since G is open, there exists a neighbourhood V of g such that $V \subseteq G$. Then, for sufficiently small $\varepsilon > 0$, we have $g_\varepsilon := g + \varepsilon \mathbf{1} \in V \subseteq G$. Hence, since $0 \leqslant g \ll g_\varepsilon$, we obtain $\langle l, g \rangle < \langle l, g_\varepsilon \rangle \leqslant \sup_{g' \in G} \langle l, g' \rangle$.

(c) From Proposition 2.2 and the definition of conormal sets it follows that a subset G of $\mathbb{R}^I_+$ is conormal if and only if it is a union of closed upper half-spaces.

One has the following well-known characterization of closed normal sets with the aid of closed lower min-type half-spaces (the proof is similar to that of [7, Proposition 5.15]):

PROPOSITION 2.3. *For a subset G of $\mathbb{R}^I_+$ the following conditions are equivalent*:

1°. *G is closed and normal.*

2°. *For each $x \in @G$ there exists $l \in \mathbb{R}^I_+$ such that*

$$\langle l, g \rangle \leqslant 1 < \langle l, x \rangle \qquad (g \in G). \tag{2.22}$$

Remark 2.3. Proposition 2.3 shows that a subset G of $\mathbb{R}^I_+$ is closed and normal if and only if it is an intersection of closed min-type lower half-spaces; in the language of abstract convex analysis, this means that G is "convex with respect to the family of all closed min-type lower half-spaces."

We recall that the *Minkowski gauge* of a closed normal set $G \subseteq \mathbb{R}^I_+$ is the function μ_G defined by

$$\mu_G(x) = \inf\{\lambda > 0 \mid x \in \lambda G\} \qquad (x \in \mathbb{R}^I_+). \tag{2.23}$$

Let us mention some well-known properties of the Minkowski gauge of a closed normal set G (see, e.g., [3, 5]).

1. $0 \leqslant \mu_G(x) \leqslant +\infty$; $\mu_G(x) = +\infty$ if and only $\{\alpha x \mid \alpha \geqslant 0\} \cap G = \{0\}$; $\mu_G(x) = 0$ if and only if $\{\alpha x \mid \alpha \geqslant 0\} \subset G$;
2. μ_G is positively homogeneous of the first degree;
3. μ_G is increasing;
4. if $\mu_G(x) < +\infty$ then $x \in \mu_G(x)\, G$;
5. $G = \{x \in \mathbb{R}^I_+ \mid \mu_G(x) \leqslant 1\}$;
6. μ_G is lower semi-continuous.

On the other hand, each function s with properties 1, 2, 3, and 6 is the Minkowski gauge of the closed normal set $G = \{x \in \mathbb{R}^I_+ \mid s(x) \leqslant 1\}$.

Let $p = \mu_G$, where G is a closed normal set.

LEMMA 2.2. *For $g' \in G$ with $\mu_G(g') = 1$ and $l \in \mathbb{R}^I_+$ the following statements are equivalent*:

1°. $l \in \partial_L \mu_G(g')$.

2°. *We have*

$$\langle l, g\rangle \leqslant 1 \qquad (g \in G), \qquad \langle l, g'\rangle = 1. \tag{2.24}$$

Proof. If 1° holds, then by (2.4) and our assumption on g', we have $\langle l, g\rangle \leqslant \mu_G(g) \leqslant 1$ for all $g \in G$ and $\langle l, g'\rangle = \mu_G(g') = 1$.

Conversely, assume 2°. Let $z \in \mathbb{R}^I_+$. If $\mu_G(z) = +\infty$, then, clearly, $\langle l, z\rangle \leqslant \mu_G(z)$. If $0 < \mu_G(z) < +\infty$, then $x := z/\mu_G(z)$ satisfies $\mu_G(x) = 1$, so $x \in G$, and therefore, by (2.24), $\langle l, x\rangle \leqslant 1$, whence $\langle l, z\rangle \leqslant \mu_G(z)$. If $\mu_G(z) = 0$, then $\lambda z \in G$ for all $\lambda > 0$, whence, by (2.24), $\langle l, \lambda z\rangle \leqslant 1$ for all $\lambda > 0$, so $\langle l, z\rangle = 0 = \mu_G(z)$. Also, by (2.24) and our assumption on g', $\langle l, g'\rangle = 1 = \mu_G(g')$. ∎

We shall take advantage of the well-known concept of weak Pareto point (see, e.g., [2]). For our study of the separation of closed normal sets and balls it will be convenient to present this concept in the following form:

DEFINITION 2.2. Let G be a closed normal set. A point $g \in G$ will be called a *weak Pareto point* (or, briefly, a *w.P. point*) of G if $(1+\lambda)\, g \notin G$ for all $\lambda > 0$, that is, if

$$G \cap \{\alpha g \mid \alpha > 1\} = \emptyset. \tag{2.25}$$

Remark 2.4. (a) *$g \in G$ is a w.P. point of G if and only if $\mu_G(g) = 1$.* Indeed, by Definition 2.2, $g \in G$ is a w.P. point of G if and only if $g \notin \frac{1}{1+\lambda} G$ for all $\lambda > 0$, which is equivalent to $\mu_G(g) = \inf_{\lambda > 0,\, g \in \lambda G} \lambda = 1$.

(b) *Each w.P. point g of G belongs to* $\mathrm{bd}_0\, G$. Indeed, if $g \in \mathrm{int}_0\, G$, then there exists an open neighbourhood V of g such that $V \subset G$. But then, for sufficiently small λ we have $(1+\lambda)\, g \in V \subset G$, and hence g is not a w.P. point of G.

(c) In general, the converse of (b) is not true (see Proposition 2.4 below). However, *if $g \gg 0$ and $g \in \mathrm{bd}_0\, G$, then g is a w.P. point of G.* Indeed, assume that $g \gg 0$ and g is not a w.P. point of G, i.e., there exists $\alpha > 1$ such that $\alpha g \in G$. Then, since $g \gg 0$, the set

$$V := \{ y \in \mathbb{R}^I_+ \mid 0 \ll y \ll \alpha g \}$$

is an open neighbourhood of g and, since G is a normal set, $V \subset G$. Thus, $g \notin \mathrm{bd}_0\, G$, which completes the proof.

Let us recall from [1] the following notion:

DEFINITION 2.3. A closed normal set G is called *regular* if $G \cap \mathrm{int}\, \mathbb{R}^I_+ \neq \emptyset$ and each ray $R_x := \{\alpha x \mid \alpha \geqslant 0\}$ with $x \neq 0$ does not intersect the boundary $\mathrm{bd}_0\, G$ of the set G (in the topological space $\mathbb{R}^I_+$) more than once; that is,

$$|R_x \cap \mathrm{bd}_0\, G| \leqslant 1 \qquad (x \in \mathbb{R}^I_+ \setminus \{0\}), \tag{2.26}$$

where $|A|$ denotes the cardinality of the set A.

Let us mention (although we shall not use this fact) that, by [1, Proposition 2.1], a closed normal set G with $\mathrm{int}_0\, G \neq \emptyset$ is regular if and only if μ_G is continuous.

PROPOSITION 2.4. *Let G be a closed normal set and let $G \cap \mathrm{int}\, \mathbb{R}^I_+ \neq \emptyset$. The following statements are equivalent*:

1°. *The set G is regular.*

2°. *Each $g \in \mathrm{bd}_0\, G$ is a w.P. point of G.*

Proof. 1° ⇒ 2°. Assume, a contrario, that 1° holds and there exists $g \in \mathrm{bd}_0\, G$ which is not a w.P. point of G, so there exists $\alpha > 1$ such that $\alpha g \in G$. We claim that $\alpha g \in \mathrm{bd}_0\, G$. Indeed, if not, then G contains an open neighbourhood $V := \{x \in \mathbb{R}^I_+ \mid (0 \leqslant) z_i < x_i < u_i (i \in I)\}$ of αg. But, then $V' := \{x \in \mathbb{R}^I_+ \mid \frac{1}{\alpha} z_i < x_i < u_i (i \in I)\}$ is an open neighbourhood of g, which is contained in G (since $V \subset G$ and since G is a normal set), so $g \in \operatorname{int}_0 G$, in contradiction to our assumption; this proves the claim that $\alpha g \in \operatorname{bd}_0 G$. However, this fact, together with $g \in \operatorname{bd}_0 G$, contradicts the regularity of G. Thus g is a w.P. point of G.

$2° \Rightarrow 1°$. Let $x \in \mathbb{R}^I_+ \setminus \{0\}$. If $R_x \cap \operatorname{bd}_0 G = \emptyset$, then (2.26) holds. If $R_x \cap \operatorname{bd}_0 G \neq \emptyset$, we may assume that $x \in R_x \cap \operatorname{bd}_0 G$. Since $x \in \operatorname{bd}_0 G$, it follows from 2° that x is a w.P. point of G, so $\lambda x \notin G$ for all $\lambda > 1$. Assume now that there exists a positive $\lambda_0 < 1$ such that $\lambda_0 x \in \operatorname{bd}_0 G$. Then, again by 2°, $\lambda_0 x$ is a w.P. point of G, whence, since $\frac{1}{\lambda_0} > 1$, we obtain $x = (1/\lambda_0)(\lambda_0 x) \notin G$, in contradiction to our assumption. Therefore $R_x \cap \operatorname{bd}_0 G = \{x\}$, so (2.26) holds. ∎

In this paper, by $\|\cdot\|$ we shall always mean the max-norm $\|\cdot\|_\infty$ of (1.4). For any $x^0 \in \mathbb{R}^I_+$ and $r' > 0$ we shall denote by $B_0(x^0, r')$ the ball in $\mathbb{R}^I_+$ with center x^0 and radius r'; that is,

$$B_0(x^0, r') = \{y \in \mathbb{R}^I_+ \mid \|x^0 - y\| \leqslant r'\}. \tag{2.27}$$

3. THE LEAST ELEMENT OF BEST APPROXIMATION

In contrast to the case of best approximation by elements of convex sets, in the case of best approximation by elements of normal sets, in the norm $\|\cdot\| = \|\cdot\|_\infty$, there exists an element of best approximation, namely the least element of best approximation, which is of special interest. Let us compute it.

Let $x^0 \in \mathbb{R}^I_+$ and $r := \operatorname{dist}(x^0, G)$. We recall (see Section 1) that $P_G(x^0) := \{g \in G \mid \|x^0 - g\| = r\}$. Let

$$I_+ := \{i \in I \mid x_i^0 > r\}, \quad I_0 := \{i \in I \mid x_i^0 = r\}, \quad I_- := \{i \in I \mid x_i^0 < r\}. \tag{3.1}$$

THEOREM 3.1. *$g^0 := \min P_G(x^0)$ (i.e., the least element g^0 of $P_G(x^0)$) exists, namely,*

$$g^0 = (x^0 - r\mathbf{1})^+; \tag{3.2}$$

that is,

$$g_i^0 = \begin{cases} x_i^0 - r & \text{if } i \in I_+ \\ 0 & \text{if } i \in I_0 \cup I_-. \end{cases} \tag{3.3}$$

Proof. We claim that

$$B_0(x^0, r) \subseteq \{ y \in \mathbb{R}^I_+ \mid y \geqslant g^0 \}. \tag{3.4}$$

Indeed, assume, a contrario, that there exists $y \in B_0(x^0, r)$ such that $y \ngeqslant g^0$. Then there exists $j \in I$ such that $y_j < g^0_j$. Hence, by (3.3), we obtain

$$r \geqslant \|x^0 - y\| \geqslant x^0_j - y_j > x^0_j - g^0_j \geqslant \max(r, x^0_j) \geqslant r,$$

which is impossible. This proves the claim (3.4).

Now take any $g \in P_G(x^0)$. Then $\|x^0 - g\| = r$, so $g \in B_0(x^0, r)$, whence, by (3.4), $g \geqslant g^0$. Hence, since G is a normal set, $g^0 \in G$. Note also that, by (3.3), we have

$$g^0 \leqslant x^0. \tag{3.5}$$

Finally, by (3.5), (3.3), and (3.1), we have

$$\|x^0 - g^0\| = \max_{i \in I} |x^0_i - g^0_i| = \max_{i \in I}(x^0_i - g^0_i) \leqslant r,$$

so $g^0 \in P_G(x^0)$, which, since we have seen above that $g \geqslant g^0$ for all $g \in P_G(x^0)$, yields the conclusion. ∎

From now on we shall use the notation g^0 for the element $\min P_G(x^0)$, without any special mention.

COROLLARY 3.1. *For the sets* (3.1) *we have*

$$I_+ = \{ i \in I \mid g^0_i > 0 \} = I(g^0), \qquad I_0 \cup I_- = \{ i \in I \mid g^0_i = 0 \}. \tag{3.6}$$

Proof. Obvious from (3.3) and (3.1). ∎

Remark 3.1. Let us observe that, while $P_G(x^0)$ need not be convex (see Example 4.2 below), $P_G(x^0)$ *is always starshaped with respect to* g^0. Indeed, for any $g' \in P_G(x^0)$ and any $0 \leqslant \lambda \leqslant 1$ we have $\lambda g^0 + (1 - \lambda)\, g' \leqslant g'$, whence $\lambda g^0 + (1 - \lambda)\, g' \in G$ (since G is a normal set), and hence

$$r \leqslant \|x^0 - \lambda g^0 - (1 - \lambda)\, g'\| \leqslant \lambda\, \|x^0 - g^0\| + (1 - \lambda)\, \|x^0 - g'\| = r,$$

so $\lambda g^0 + (1 - \lambda)\, g' \in P_G(x^0)$.

4. MIN-TYPE SEPARATION OF A CLOSED NORMAL SET G AND A BALL

Let G be a closed normal subset of $\mathbb{R}^I_+$ and let $x^0 \in @G$. Our aim will be to give, in the next section, necessary and sufficient conditions for a given $g' \in G$ in order to have $g' \in P_G(x^0)$. To this end, suggested by the case of convex sets, we want to use separation conditions. We know from Proposition 2.3 that G can be separated from each outside point $x^0 \in \mathbb{R}^I_+ \setminus G$ by a min-type function l, but this will not be sufficient for our purpose; as in the convex case, we need separation of G and a ball (in the norm $\|\cdot\| = \|\cdot\|_\infty$) centered at x^0. In this Section we shall study "min-type separation" of a closed normal set G and a ball $B_0(x^0, r')$, where $r' := \|x^0 - g'\|$ for some $g' \in G$, i.e., the problem of conditions under which there exists a vector $l \in \mathbb{R}^I_+ \setminus \{0\}$ such that

$$\langle l, g\rangle \leqslant \langle l, y\rangle \qquad (g \in G,\ y \in B_0(x^0, r')). \tag{4.1}$$

We shall be interested especially in the particular case of min-type separation of a closed normal set G and the ball $B_0(x^0, r)$, where $r := \mathrm{dist}(x^0, G) = \|x^0 - g^0\|$, with $g^0 := \min P_G(x^0)$.

We can present (4.1) in the following equivalent form: There exist $l \in \mathbb{R}^I_+ \setminus \{0\}$ and a number $\gamma \geqslant 0$ such that

$$\langle l, g\rangle \leqslant \gamma \leqslant \langle l, y\rangle \qquad (g \in G,\ y \in B_0(x^0, r')). \tag{4.2}$$

If $\gamma > 0$, we can consider the vector l/γ instead of l; hence, in this case we may assume that $\gamma = 1$. Thus we can consider the following two kinds of separation:

1°. There exists $l \in \mathbb{R}^I_+ \setminus \{0\}$ such that

$$\langle l, g\rangle \leqslant 1 \leqslant \langle l, y\rangle \qquad (g \in G,\ y \in B_0(x^0, r')). \tag{4.3}$$

2°. There exists $l \in \mathbb{R}^I_+ \setminus \{0\}$ such that

$$\langle l, g\rangle = 0 \leqslant \langle l, y\rangle \qquad (g \in G,\ y \in B_0(x^0, r')). \tag{4.4}$$

Remark 4.1. (a) Clearly, *case* 2° *holds if and only if the first part of* (4.4) *holds, i.e.,* $\langle l, g\rangle = 0$ $(g \in G)$. *This, in turn, is equivalent to* $G \cap \mathbb{R}^I_{++} = \emptyset$. Indeed, if $g \in G \cap \mathbb{R}^I_{++}$, then $\langle l, g\rangle = \min_{i \in I(l)} l_i g_i > 0$ for all $l \neq 0$, in contradiction to the first part of (4.4); thus, $G \cap \mathbb{R}^I_{++} = \emptyset$. On the other hand, if $G \cap \mathbb{R}^I_{++} = \emptyset$, then for each $g \in G$ there exists $i \in I$ such that $g_i = 0$. Let $l = \mathbf{1}$. Then, clearly, $\langle l, g\rangle = 0$ for all $g \in G$.

(b) We shall not be interested in case 2°, since for any function $l \in \mathbb{R}^I_+ \setminus \{0\}$ as in case 2°, formula (4.4) remains also valid if we replace $B_0(x^0, r')$ by any other subset of $\mathbb{R}^I_+$, that is, l separates G and any other subset of $\mathbb{R}^I_+$, so (4.4) is not useful for the problem of best approximation of x^0 by the elements of G. Therefore, hereafter we shall consider only separation of the form (4.3).

(c) For any $l \in \mathbb{R}^I_+$ satisfying (4.3) we have $l \neq 0$ (since $\langle 0, y\rangle = 0 \ngeqslant 1$).

Let us give now a necessary condition for separability (4.3), which, in an important particular case, is also sufficient.

THEOREM 4.1. *Let G be a closed normal set, $x^0 \in @G$, $g' \in G$, and $r' = \|x^0 - g'\|$. Let us consider the following statements*:

1°. *There exists $l \in \mathbb{R}^I_+$ satisfying* (4.3).

2°. *g' is a weak Pareto point of the set G.*

Then $1° \Rightarrow 2°$.

In the particular case when $g' = g^0 := \min P_G(x^0)$, the converse is also true; i.e., the following statements are equivalent:

1′. *There exists $l \in \mathbb{R}^I_+$ satisfying*

$$\langle l, g \rangle \leqslant 1 \leqslant \langle l, y \rangle \qquad (g \in G,\ y \in B_0(x^0, r)), \tag{4.5}$$

where $r = \min_{g \in G} \|x^0 - g\|$.

2′. *$g^0 := \min P_G(x^0)$ is a weak Pareto point of the set G.*

Moreover, if there exists $l \in \mathbb{R}^I_+$ satisfying (4.3), *then the same l also satisfies* (4.5), *so* $1° \Rightarrow 1'$. *Also, if $2'$ holds, then we have* (4.5) *with $l := 1/g^0$.*

Proof. Assume 1°. Then, since $g' \in G \cap B_0(x^0, r')$, from (4.3) it follows that $\langle l, g' \rangle = 1$. If g' is not a w.P. point, so there exists $\alpha > 1$ such that $\alpha g' \in G$, then $\langle l, \alpha g' \rangle = \alpha \langle l, g' \rangle = \alpha > 1$, which contradicts (4.3). This proves that $1° \Rightarrow 2°$, and hence, in particular, $1' \Rightarrow 2'$. Also, the implication that $1° \Rightarrow 1'$, with the same l, follows from the fact that $B_0(x^0, r) \subseteq B_0(x^0, r')$ (since $r \leqslant r'$; actually, from Theorem 5.1 below it will follow that $r' = r$).

Conversely, assume now $2'$. We shall show that (4.5) holds with $l = 1/g^0$, which will complete the proof. If $y \in B_0(x^0, r)$, then, by (3.4), we have $y \geqslant g^0$, so $\langle l, y \rangle \geqslant \langle l, g^0 \rangle = 1$. We now prove the first part of (4.5). Since g^0 is a w.P. point, by Remark 2.4(a) we have $\mu_G(g^0) = 1$. Hence, by (2.7), $l = 1/g^0 = \mu_G(g^0)/g^0 \in \partial_L \mu_G(g^0)$. Consequently, by Lemma 2.2, for each $g \in G$ we have $\langle l, g \rangle \leqslant 1$. ∎

In general, the converse implication $2° \Rightarrow 1°$ is not true, as shown by

EXAMPLE 4.1. Let $I = \{1, 2\}$, $G = \{(0, g_2) \mid 0 \leqslant g_2 \leqslant 3\}$, $x^0 = (1, 3)$, and $g' = (0, 3)$. Then g' is a w.P. point of G, but $r' = \|x^0 - g'\| = 1 = r$, and $g^0 = (0, 2)$ is not a w.P. point of G, whence, by Theorem 4.1, implication $1' \Rightarrow 2'$, there exists no $l \in \mathbb{R}^I_+$ satisfying (4.5) (which coincides with (4.3), since $r' = r$).

In connection with the last statement of Theorem 4.1 let us note that, even when there exists $l \in \mathbb{R}^I_+$ satisfying (4.3) (hence g' is a weak Pareto point of the set G), $1/g'$ need not separate G and $B_0(x^0, r')$, or even G and x^0, as shown by

EXAMPLE 4.2. Let $I=\{1, 2\}$, $G:=\{g\in\mathbb{R}^2_+ \mid \min(g_1, g_2)\leqslant 1\}$, and $x^0=(2, 2)$. Then $P_G(x^0)$ is the union of two segments, $[(1, 1), (3, 1)]\cup[(1, 1), (1, 3)]$, and for $g':=(1, 3)$ we have $r'=\|x^0-g'\|=1$ and $g'\in P_G(x^0)$. Also, $g^0:=\min P_G(x^0)=(1, 1)$, so $\|x^0-g^0\|=1$, and, clearly, $1/g^0=(1, 1)$ separates G and $B_0(x^0, 1)=B_0(x^0, r')$ (this follows also from the second part of Theorem 4.1, since g^0 is a w.P. point of G). However, $1/g'=(1, \frac{1}{3})$ does not even separate G and x^0. Indeed, although

$$\left\langle \frac{1}{g'}, g \right\rangle = \min\left(g_1, \frac{1}{3} g_2\right) \leqslant \min(g_1, g_2) \leqslant 1 \qquad (g\in G), \tag{4.6}$$

we have

$$\left\langle \frac{1}{g'}, x_0 \right\rangle = \min\left(2, \frac{2}{3}\right) = \frac{2}{3} < 1.$$

COROLLARY 4.1. *If there exists* $l\in\mathbb{R}^I_+$ *satisfying* (4.3), *then*

$$R_{g'} \nsubseteq G, \tag{4.7}$$

where $R_{g'}=\{\alpha g' \mid \alpha>0\}$.

Proof. This follows from Theorem 4.1, since every w.P. point g' satisfies (4.7). ∎

Some other necessary conditions for separability (4.3) are given in

PROPOSITION 4.1. *Let* G *be a closed normal set,* $x^0\in @G$, $g'\in G$, *and* $r'=\|x^0-g'\|$. *If there exists* $l\in\mathbb{R}^I_+$ *satisfying* (4.3), *then for each such* l *we have*

$$\{g\in G \mid g_i<g'_i\ (i\in I(l))\} \cap B_0(x^0, r')=\emptyset. \tag{4.8}$$

Also, then

$$\lambda g' \notin B_0(x^0, r') \qquad (0\leqslant\lambda<1), \tag{4.9}$$

$$\|x^0\| > r'. \tag{4.10}$$

Proof. Assume, a contrario, that there exists an element $\bar{g}$ in the intersection (4.8). Then, since $\bar{g}_i<g'_i\ (i\in I(l))$, from (4.3) we obtain $\langle l, \bar{g}\rangle < \langle l, g'\rangle \leqslant 1$. On the other hand, since $\bar{g}\in B_0(x^0, r')$, by (4.3) we have $\langle l, \bar{g}\rangle \geqslant 1$, a contradiction.

The proof of (4.9) is similar: Let $0<\lambda<1$. Then, since G is a normal set, we have $\lambda g' \in G$, and hence, by the first part of (4.3), $\langle l, \lambda g' \rangle = \lambda < 1$. If $\lambda g' \in B_0(x^0, r')$, then, by the second part of (4.3), $\langle l, \lambda g' \rangle \geqslant 1$, a contradiction.

Finally, (4.10) is the particular case $\lambda = 0$ of (4.9). ∎

PROPOSITION 4.2. *Under the assumptions of Theorem* 4.1:

(a) *For each* $l \in \mathbb{R}^I_+$ *satisfying* (4.3) *and each* $g' \in G$ *there exists an index* $j \in I(l)$ *such that*

$$g'_j \leqslant g^0_j. \tag{4.11}$$

(b) *For each* $l \in \mathbb{R}^I_+$ *satisfying* (4.3) *and each* $g' \in P_G(x^0)$ *there exists an index* $j \in I(l)$ *such that*

$$g'_j = g^0_j. \tag{4.12}$$

Proof. (a) Since $g^0 \in G \cap B_0(x^0, r) \subseteq G \cap B_0(x^0, r')$, by (4.3) we have $\langle l, g^0 \rangle = 1$. Assume now, a contrario, that there exists no index $j \in I(l)$ such that (4.11) holds, i.e., that $g'_i > g^0_i$ for all $i \in I(l)$. Then, by the first part of (4.3), we obtain $1 \geqslant \langle l, g' \rangle > \langle l, g^0 \rangle = 1$, which is impossible.

(b) This follows from part (a) and the fact that $g^0 \leqslant g'$ for all $g' \in P_G(x^0)$. ∎

Remark 4.2. In Remark 3.1 it has been observed that for each $g' \in P_G(x^0)$ the segment $\{\lambda g^0 + (1-\lambda)\, g' \mid 0 \leqslant \lambda \leqslant 1\}$ is contained in $P_G(x^0)$. In this connection, note that any index j as in (4.12) is common to all points of this segment, since (4.12) implies $\lambda g^0_j + (1-\lambda)\, g'_j = g^0_j$.

Now we can give the following sufficient condition for separability (4.5), for each $x^0 \in @G$ such that $g^0 = \min P_G(x^0) \gg 0$.

PROPOSITION 4.3. *If* $g^0 \gg 0$, *then there exists* $l \in \mathbb{R}^I_+$ *satisfying* (4.5).

Proof. By $g^0 \in \mathrm{bd}_0\, G$ and Remark 2.4(c), g^0 is a w.P. point of G. Hence, by Theorem 4.1, the conclusion follows. ∎

Let us also give the following sufficient condition for separability (4.5), for each $x^0 \in @G$:

THEOREM 4.2. *If G is a regular closed normal set, then for each $x^0 \in @G$ there exists an element $l \in \mathbb{R}^I_+$ satisfying* (4.5).

Proof. It is well known (and easy to see) that $P_G(x^0) \subseteq \mathrm{bd}_0\, G$, so $g^0 \in \mathrm{bd}_0\, G$. Hence, by Proposition 2.4, g^0 is a w.P. point of G. Consequently, by Theorem 4.1, there exists $l \in \mathbb{R}^I_+$ satisfying (4.5). ∎

The following example shows that the separability (4.5) need not hold for non-regular sets G, even when $G \cap \operatorname{int} \mathbb{R}^I_+ \neq \emptyset$:

EXAMPLE 4.3. Let $I = \{1, 2\}$ and $G = G_1 \cup G_2$, where $G_1 = \{(g_1, g_2) \mid 0 \leqslant g_1 \leqslant 1,\ 0 \leqslant g_2 \leqslant 1\}$ and $G_2 = \{(0, g_2) \mid 1 \leqslant g_2 \leqslant 3\}$, and let $x^0 = (1, 3)$. Then $P_G(x^0) = \{(0, g_2) \mid 2 \leqslant g_2 \leqslant 3\}$ and hence the least element g^0 of the set $P_G(x^0)$ is $g^0 = (0, 2)$. Clearly, the intersection $R_{g^0} \cap \operatorname{bd}_0 G$ coincides with the segment G_2, so the set G is not regular. Also, g^0 is not a w.P. point of G (in fact, $\mu_G(g^0) = 2/3$), and hence, by Theorem 4.1, there exists no element $l \in \mathbb{R}^I_+$ satisfying (4.5).

Remark 4.3. Assume that G is a closed normal set with $G \cap \mathbb{R}^I_{++} = \emptyset$, for which there exists a set of indices $I' \subset I$, $I' \neq I$, such that $G \subset \mathbb{R}^{I'}_+$. If G is regular in $\mathbb{R}^{I'}_+$ and $x^0 \in \mathbb{R}^{I'}_+ \cap @G$, then, since G is closed and normal also in $\mathbb{R}^{I'}_+$, by Theorem 4.2 there exists $l' \in \mathbb{R}^{I'}_+$ which min-type separates G and the projection of $B_0(x^0, r)$ onto $\mathbb{R}^{I'}_+ \times \{0\}$ (defined as in (4.16) below, with I_+ and $I_0 \cup I_-$ replaced by I' and its complement in I, respectively) and hence the vector $l \in \mathbb{R}^I_+$ defined by

$$l_i = \begin{cases} l'_i & \text{if } i \in I' \\ 0 & \text{if } i \notin I' \end{cases} \tag{4.13}$$

will satisfy $I(l) = I'(l')$ and it will separate G and the ball $B_0(x^0, r)$. However, if G is regular in $\mathbb{R}^{I'}_+$ and $x^0 \notin \mathbb{R}^{I'}_+$, then min-type separability (4.5) of G and the ball $B_0(x^0, r)$ need not hold, as shown by Example 4.1, in which $I' = \{2\}$ and G is regular in $\mathbb{R}^{I'}_+$.

Let us give now another sufficient condition for min-type separability (4.5).

THEOREM 4.3. *Let G be a closed normal set and $x^0 \in @G$, such that*

$$R_{g^0} \not\subseteq G, \tag{4.14}$$

$$I_0 = \emptyset, \tag{4.15}$$

where $g^0 := \min P_G(x^0)$, $R_{g^0} := \{\alpha g^0 \mid \alpha \geqslant 0\}$ *and* $I_0 := \{i \in I \mid x^0_i = r\}$ (*see* (3.1)). *Then there exists an element* $l \in \mathbb{R}^I_+$ *satisfying* (4.5).

Proof. For each $x \in \mathbb{R}^I_+ = \mathbb{R}^{I_+}_+ \times \mathbb{R}^{I_-}_+$ (with I_+ and I_- of (3.1)), let us denote by $\pi(x)$ the projection of x onto $\mathbb{R}^{I_+}_+ \times \{0\}$, defined by

$$\pi(x)_i = \begin{cases} x_i & \text{if } i \in I_+ \\ 0 & \text{if } i \in I_-. \end{cases} \tag{4.16}$$

Then, since G is a closed normal set, so is $\pi(G) := \{\pi(g) \mid g \in G\}$ in $\pi(\mathbb{R}^I_+)$. Furthermore, by (3.3),

$$\|\pi(x^0) - \pi(g^0)\| = \max_{i \in I_+}(x_i^0 - g_i^0) = r. \tag{4.17}$$

We claim that

$$\pi(g^0) \in P_{\pi(G)}(\pi(x^0)), \qquad \min_{\pi(g) \in \pi(G)} \|\pi(x^0) - \pi(g)\| = r. \tag{4.18}$$

Indeed, by (4.17) we have $\min_{\pi(g) \in \pi(G)} \|\pi(x^0) - \pi(g)\| \leqslant r$. Assume now, a contrario, that there exists $\pi(g) \in \pi(G)$ such that $\|\pi(x^0) - \pi(g)\| < r$. Define $\bar{g} \in \mathbb{R}^I_+$ by

$$\bar{g}_i = \begin{cases} g_i & \text{if} \quad i \in I_+ \\ 0 & \text{if} \quad i \in I_-. \end{cases} \tag{4.19}$$

Then $0 \leqslant \bar{g} \leqslant g$, whence, since G is a normal set, $\bar{g} \in G$. Also, by (4.19), (4.16), (4.15), $\|\pi(x^0) - \pi(g)\| < r$, (3.3), and (3.1), we have

$$\|x^0 - \bar{g}\| = \max\{\max_{i \in I_+} |\pi(x^0)_i - \pi(g)_i|, \max_{i \in I_-} x_i^0\} < r = \min_{g \in G} \|x^0 - g\|,$$

which is impossible. This proves the claim (4.18). Hence, by $g^0 = \min P_G(x^0)$ and (4.16), it follows that

$$\pi(g^0) = \min P_{\pi(G)}(\pi(x^0)). \tag{4.20}$$

Now we shall show that

$$\mu_{\pi(G)}(\pi(g^0)) = 1. \tag{4.21}$$

Indeed, by $\pi(g^0) \in \pi(G)$ we have $0 \leqslant \mu_{\pi(G)}(\pi(g^0)) \leqslant 1$. Also, by the assumption (4.14), $\mu_{\pi(G)}(\pi(g^0)) > 0$. Assume now, a contrario, that $\mu_{\pi(G)}(\pi(g^0)) < 1$. Then, since $\mu_{\pi(G)}$ is lower semi-continuous at $\pi(g^0)$, there exists a neighbourhood V of $\pi(g^0)$ such that $\mu_{\pi(G)}(x) < 1$ for all $x \in V$, whence $V \subseteq \pi(G)$. For $0 < \lambda < 1$, let us define

$$x'_\lambda := \lambda \pi(x^0) + (1 - \lambda)\, \pi(g^0). \tag{4.22}$$

Then, for sufficiently small $\lambda > 0$, $\|x'_\lambda - \pi(g^0)\| = \lambda\, \|\pi(x^0) - \pi(g^0)\|$ is near to 0, so $x'_\lambda \in V \subseteq \pi(G)$. Also, for any such λ, by (4.22) and (4.17) we have

$$\|\pi(x^0) - x'_\lambda\| = (1 - \lambda)\, \|\pi(x^0) - \pi(g^0)\| = (1 - \lambda)\, r < r,$$

which contradicts (4.18). This proves (4.21). Hence, by Remark 2.4(a), $\pi(g^0)$ is a w.P. point of $\pi(G)$. Consequently, by Theorem 4.1, the element $l' = 1/\pi(g^0)$ separates the set $\pi(G)$ and the ball $B_0(\pi(x^0), r) \cap (\mathbb{R}^{I_+}_+ \times \{0\})$; that is,

$$\langle l', \pi(g)\rangle \leqslant 1 \leqslant \langle l', \pi(y)\rangle \quad (\pi(g) \in \pi(G), \pi(y) \in B_0(\pi(x^0), r)). \tag{4.23}$$

Now define $l \in \mathbb{R}^I_+$ by

$$l_i := \begin{cases} l'_i & \text{if} \quad i \in I_+ \\ 0 & \text{if} \quad i \in I_-. \end{cases} \tag{4.24}$$

Then $\langle l, g\rangle = \langle l', \pi(g)\rangle \leqslant 1$ for all $g \in G$. Also, since $\|x^0 - y\| \leqslant r$ implies that $\|\pi(x^0) - \pi(y)\| = \max_{i \in I_+} |\pi(x^0)_i - \pi(y)_i| \leqslant r$, we have $\langle l, y\rangle = \langle l', \pi(y)\rangle \geqslant 1$ for all $y \in B_0(x^0, r)$. ∎

Remark 4.4. By Corollary 4.1, condition (4.14) is a necessary condition for separability (4.5). However, it is not a sufficient condition; i.e., the assumption (4.15) cannot be omitted from Theorem 4.3, as shown by Example 4.1.

The following theorem gives a necessary condition for a given $l \in \mathbb{R}^I_+$ to satisfy (4.3), which, in the particular case where $g' = g^0$, $r' = r$, is a necessary and sufficient condition for a given $l \in \mathbb{R}^I_+$ to satisfy (4.5), or, in other words, a description of all $l \in \mathbb{R}^I_+$ that separate G and $B_0(x^0, r)$ in the sense (4.5).

THEOREM 4.4. *Let G be a closed normal set, $x^0 \in @G$, $g' \in G$, and $r' := \|x^0 - g'\|$. For $l \in \mathbb{R}^I_+$ let us consider the following statements*:

1°. *l separates G and $B_0(x^0, r')$ in the sense* (4.3).

2°. *We have*

$$I(l) \subseteq I_+, \tag{4.25}$$

$$\langle l, g'\rangle = 1, \tag{4.26}$$

$$\frac{1}{l} \text{ is a weak Pareto point of } G. \tag{4.27}$$

Then 1° ⇒ 2°.

In the particular case where $g' = g^0 := \min P_G(x^0)$, the converse is also true; *moreover, the following statements are equivalent*:

1′. *l separates G and* $B_0(x^0, r)$ *in the sense* (4.5), *where* $r := \|x^0 - g^0\|$.

2′. *We have* (4.27), *and*

$$\langle l, g^0 \rangle = 1. \tag{4.28}$$

Proof. Assume 1°. Then, since $g' \in G \cap B_0(x^0, r')$, by (4.3) we have (4.26). Hence, by 1° and Lemma 2.2, we obtain $l \in \partial_L \mu_G(g')$ and thus, by Proposition 2.1, $\mu_G(\frac{1}{l}) = 1$, which, by Remark 2.4(a), is equivalent to (4.27). Finally, since $g^0 \in G \cap B_0(x^0, r) \subseteq G \cap B_0(x^0, r')$, by (4.3) we have (4.28), and hence $I(l) \subseteq I(g^0)$, which, together with (3.6), yields (4.25). This proves that $1° \Rightarrow 2°$, and hence, in particular, $1' \Rightarrow 2'$.

Conversely, assume now 2′. Then, by (4.27) and Remark 2.4(a), we have $\mu_G(\frac{1}{l}) = 1$. Furthermore, by (4.28) and $g^0 \in G$, we have $\langle l, g^0 \rangle = 1 \geqslant \mu_G(g^0)$. Hence, by Proposition 2.1, $l \in \partial_L \mu_G(g^0)$, and therefore, by (2.4), we obtain $\langle l, g \rangle \leqslant \mu_G(g) \leqslant 1$ $(g \in G)$. Finally, by (3.4) and (4.28), we obtain the second part of (4.5). ▌

Finally, let us make some remarks about connections between the separation of G and the ball $B_0(x^0, r)$, in the sense (4.5), and the separation of G and x^0, in the "normalized min-type sense"

$$\langle l, g \rangle \leqslant 1 \leqslant \langle l, x^0 \rangle \qquad (g \in G). \tag{4.29}$$

First, as we have seen in the above, while a closed normal set G and an outside point x^0 can always be separated by some $l \in \mathbb{R}^I_+$, in the sense (4.29) (see Section 2), G and $B_0(x^0, r)$ can be separated if and only if g^0 is a w.P. point of G. However, even when g^0 is a w.P. point of G, a given $l \in \mathbb{R}^I_+$ which separates G and x_0 need not separate G and $B_0(x^0, r)$, as shown by

Example 4.4. Let I, G, and x^0 be as in Example 4.2, and let $l = (1, \frac{2}{3})$. Then l separates G and x^0 (since $\langle l, g \rangle = \min(g_1, \frac{2}{3}g_2) \leqslant \min(g_1, g_2) \leqslant 1$ for all $g \in G$ and $\langle l, x^0 \rangle = \min(x^0_1, \frac{2}{3}x^0_2) = \min(2, \frac{4}{3}) = \frac{4}{3} \geqslant 1$), but l does not separate G and $B_0(x^0, r)$ (since for $g^0 = (1, 1) \in B_0(x^0, r)$ we have $\langle l, g^0 \rangle = \min(1, \frac{2}{3}) < 1$).

By Theorem 4.1, if there exists $l \in \mathbb{R}^I_+$ which separates G and $B_0(x^0, r)$ (or, equivalently, if g^0 is a w.P. point of G), then $1/g^0$, too, separates G and $B_0(x^0, r)$, whence also G and x^0. However, in the general case $1/g^0$ need not separate G and x^0 (although, by Section 2, G and x^0 can be separated by some other l). Indeed, this is shown by

Example 4.5. Let I, G, x^0, and $g' = (0, 3) \in G$ be as in Example 4.1. Then $g^0 = (0, 2)$, and $1/g^0 = (0, \frac{1}{2})$ does not separate G and x^0, since $\langle 1/g^0, g' \rangle = \frac{3}{2} > 1$.

5. CHARACTERIZATIONS OF NEAREST POINTS IN CLOSED NORMAL SETS

The following theorem gives a sufficient condition for a given $g' \in G$ to be a nearest point to x^0.

THEOREM 5.1. *Let G be a closed normal set, $x^0 \in @G$, $g' \in G$, and $r' := \|x^0 - g'\|$. If there exists $l \in \mathbb{R}^I_+$ which (min-type) separates G and $B_0(x^0, r')$, i.e., satisfying* (4.3), *then $g' \in P_G(x^0)$. Moreover, if* (4.3) *holds with $l = 1/g'$, then $g' = g^0 := \min P_G(x^0)$.*

Proof. By (4.3) and Theorem 4.4 we have (4.25). Consequently, by (4.25) and (3.1),

$$I(l) \subseteq I_+, \qquad I(l) \cap (I_0 \cup I_-) = \emptyset \tag{5.1}$$

Assume now, a contrario, that $g' \notin P_G(x^0)$, i.e., $r' > r := \|x^0 - g^0\|$. We shall construct an element $\bar{g} \in \mathbb{R}^I_+$ with the following properties:

$$\bar{g} \in G \cap B_0(x^0, r'), \tag{5.2}$$

$$\bar{g}_i < g^0_i \qquad (i \in I(l)). \tag{5.3}$$

This will imply a contradiction, completing the proof of $g' \in P_G(x^0)$. Indeed, by (4.3) and $g^0 \in G \cap B_0(x^0, r')$, there holds $\langle l, g^0 \rangle = 1$, whence, by (5.3), $\langle l, \bar{g} \rangle < \langle l, g^0 \rangle = 1$. On the other hand, by (4.3) and (5.2), we have $\langle l, \bar{g} \rangle = 1$, a contradiction.

We claim that there exists $\lambda < 0$ such that

$$0 < \lambda x^0_i + (1-\lambda)\, g^0_i < g^0_i \qquad (i \in I_+), \tag{5.4}$$

$$|x^0_i - (\lambda x^0_i + (1-\lambda)\, g^0_i)| < r' \qquad (i \in I_+); \tag{5.5}$$

note that $\lambda x^0 + (1-\lambda)\, g^0$ is a point beyond g^0 on the ray starting in x^0 and going through g^0, but it need not belong to $\mathbb{R}^I_+$.

Proof of the Claim. Let $i \in I_+$ be arbitrary. The first inequality of (5.4) means that $\lambda(x^0_i - g^0_i) > -g^0_i$, and thus, by (3.6) and (3.3), it is enough to choose $0 > \lambda > -g^0_i/r$. The second inequality of (5.4) means that $\lambda x^0_i + (1-\lambda)\, g^0_i - g^0_i = \lambda(x^0_i - g^0_i) < 0$, which holds for all $\lambda < 0$ (by (3.3)). On the other hand, for any $\lambda < 0$ we have

$$|x^0_i - (\lambda x^0_i + (1-\lambda)\, g^0_i)| = (1-\lambda)\, |x^0_i - g^0_i| \leqslant (1-\lambda)\, r, \tag{5.6}$$

which will be $<r'$ whenever $0>\lambda>1-r'/r$ (here we use the assumption that $r'/r>1$). Consequently, for any λ such that

$$0>\lambda>\max\left\{\max_{i\in I_+}\left(-\frac{g_i^0}{r}\right),\, 1-\frac{r'}{r}\right\}, \tag{5.7}$$

we shall have (5.4) and (5.5). This proves the claim.

Finally, choose such a $\lambda<0$ and define $\bar{g}\in\mathbb{R}^I$ by

$$\bar{g}_i := \begin{cases} \lambda x_i^0+(1-\lambda)\, g_i^0 & \text{if } i\in I_+ \\ 0 & \text{if } i\in I_0\cup I_-. \end{cases} \tag{5.8}$$

We shall show that $\bar{g}$ satisfies (5.2) and (5.3). By (5.8) and (5.4) we have $0\leqslant\bar{g}\leqslant g^0$, and hence, since G is a normal set, $\bar{g}\in G$. Furthermore, for $i\in I_0\cup I_-$, by (5.8) and (3.1) we have $x_i^0-\bar{g}_i=x_i^0\leqslant r<r'$, which together with (5.5) and (5.8) proves that $\bar{g}\in B_0(x^0, r')$. Thus, (5.2) holds. Finally, let $i\in I(l)$. Then, by (5.1) and (5.8), we have $i\in I_+$ and $\bar{g}_i=\lambda x_i^0+(1-\lambda)\, g_i^0$, and hence, by (5.4), we obtain (5.3). This completes the proof of $g'\in P_G(x^0)$.

Finally, assume that (4.3) holds with $l'=1/g'$, that is,

$$\left\langle\frac{1}{g'}, g\right\rangle\leqslant 1\leqslant\left\langle\frac{1}{g'}, y\right\rangle \qquad (g\in G,\ y\in B_0(x^0, r')). \tag{5.9}$$

Then, by the above, $g'\in P_G(x^0)$. On the other hand, since $g^0\in P_G(x^0)$, we have $\|x^0-g^0\|=r\leqslant r'$, so $g^0\in B_0(x^0, r')$, whence, by (5.9), $\langle 1/g', g^0\rangle=\min_{i\in I(g')} g_i^0/g'_i\geqslant 1$. Thus, $g_i^0\geqslant g'_i$ for all $i\in I(g')$, whence also for all i; that is, $g^0\geqslant g'$. Hence, by $g'\in P_G(x^0)$ and the definition of g^0, we obtain $g'=g^0$. ∎

Remark 5.1. (a) As shown, e.g., by Example 4.1, the condition of separability of Theorem 5.1 is not necessary for $g'\in P_G(x^0)$.

(b) In the particular case where

$$x^0\geqslant r'\mathbf{1}, \tag{5.10}$$

one can give the following simpler proof of Theorem 5.1: By $x^0-r'\mathbf{1}\in B_0(x^0, r')$ and the second part of (4.3), we have

$$\frac{1}{l}\leqslant x^0-r'\mathbf{1}. \tag{5.11}$$

Assume now, a contrario, that $r=\|x^0-g^0\|<\|x^0-g'\|=r'$. Then, by $x^0-g^0\leqslant r\mathbf{1}$ and (5.11),

$$g^0\geqslant x^0-r\mathbf{1}=x^0-r'\mathbf{1}+(r'-r)\,\mathbf{1}\geqslant\frac{1}{l}+(r'-r)\,\mathbf{1},$$

and hence, by $r' > r$, we obtain $g_i^0 > 1/l_i$ $(i \in I(l))$, which contradicts the first part of (4.3),

Let us pass now to necessary and sufficient conditions.

THEOREM 5.2. *For an element $g' \in G$ the following statements are equivalent*:

1°. $g' \in P_G(x^0)$ *and* $g^0 := \min P_G(x^0)$ *is a weak Pareto point of* G.

2°. *There exists* $l \in \mathbb{R}_+^I$ *which separates* G *and* $B_0(x^0, r')$ (*where* $r' := \|x^0 - g'\|$), *in the sense* (4.3).

Proof. Assume 1°. Then, by $g' \in P_G(x^0)$, we have $r' = r$. Hence, since g^0 is a weak Pareto point of G, by Theorem 4.1 there exists $l \in \mathbb{R}_+^I$ which separates G and $B_0(x^0, r) = B_0(x^0, r')$.

Conversely, assume now 2°. Then, by Theorem 5.1 above, we have $g' \in P_G(x^0)$. Furthermore, since $B_0(x^0, r) \subseteq B_0(x^0, r')$, l also separates G and $B_0(x^0, r)$, and hence, by Theorem 4.1, g^0 is a weak Pareto point of G. ∎

Remark 5.2. It would be desirable to find characterizations of the points $g' \in P_G(x^0)$ which did not involve g^0. However, in 1° above the condition that g^0 is a weak Pareto point of G cannot be replaced by the condition that g' is a weak Pareto point of G, as shown by Example 4.1 above, in which $g' \in P_G(x^0)$ and g' is a weak Pareto point of G, but there exists no $l \in \mathbb{R}_+^I$ which separates G and $B_0(x^0, r')$, in the sense (4.3).

One can obtain such characterizations as desired in Remark 5.2, by using sufficient conditions for g^0 to be a weak Pareto point of G that do not involve explicitly g^0. For example, we have

COROLLARY 5.1. *Let G be a regular closed normal set. For an element $g' \in G$ the following statements are equivalent*:

1°. $g' \in P_G(x^0)$.

2°. *There exists* $l \in \mathbb{R}_+^I$ *which separates* G *and* $B_0(x^0, r')$ (*where* $r' := \|x^0 - g'\|$), *in the sense* (4.3).

Proof. This follows from Theorem 5.2 and the proof of Theorem 4.2. ∎

6. THE DISTANCE TO A MIN-TYPE HYPERPLANE, A LOWER MIN-TYPE HALF-SPACE, AND A NORMAL SET

Let $l = (l_1, ..., l_n) \in \mathbb{R}_+^I$, and let us consider the upper min-type half-space

$$D := \{x \in \mathbb{R}_+^I \mid \langle l, x \rangle \geqslant 1\} = \{x \in \mathbb{R}_+^I \mid \min_{i \in I(l)} l_i x_i \geqslant 1\}. \tag{6.1}$$

Furthermore, let $x^0 = (x_1^0, ..., x_n^0) \in \mathrm{int}_0\, D = \{x \in \mathbb{R}_+^I \mid \min_{i \in I(l)} l_i x_i > 1\}$; that is,

$$l_i x_i^0 > 1 \quad (i \in I(l)), \qquad x_i^0 \geqslant 0 \quad (i \notin I(l)). \tag{6.2}$$

In order to obtain an explicit formula for $\mathrm{dist}(x^0, \mathrm{bd}_0\, D) = \mathrm{dist}(x^0, @D) = \mathrm{dist}(x^0, \mathrm{bd}_0\, D \cup @D)$, where $@D := \mathbb{R}_+^I \setminus D$, the complement of D in $\mathbb{R}_+^I$ (see Lemma 2.1), let us first note the following simple fact:

Lemma 6.1. Let $c \geqslant 0$, $1 \leqslant j \leqslant n$, $U_j = \{u \in \mathbb{R}_+^I \mid u_j = c_j\}$, and $x^0 \in \mathbb{R}_+^I$. Then

$$\mathrm{dist}(x^0, U_j) = |x_j^0 - c_j|. \tag{6.3}$$

Proof. For any $u \in U_j$ we have $u_j = c_j$, whence

$$\|x^0 - u\| = \max_{i \in I} |x_i^0 - u_i| \geqslant |x_j^0 - c_j|.$$

On the other hand, for $u^0 := (x_1^0, ..., x_{j-1}^0, c_j, x_{j+1}^0, ..., x_n^0)$ we have $u^0 \in U_j$ and

$$\|x^0 - u^0\| = \max(0, |x_j^0 - c_j|) = |x_j^0 - c_j|. \quad \blacksquare$$

THEOREM 6.1. *For D and x^0 as in* (6.1) *and* (6.2), *we have*

$$\mathrm{dist}(x^0, \mathrm{bd}_0\, D) = \min_{i \in I(l)} \left(x_i^0 - \frac{1}{l_i} \right). \tag{6.4}$$

Proof. By (6.1), we have

$$D = \bigcap_{i \in I} D_i, \tag{6.5}$$

where

$$D_i := \begin{cases} \{x \in \mathbb{R}_+^I \mid l_i x_i \geqslant 1\} & \text{if} \quad i \in I(l) \\ \mathbb{R}_+^I & \text{if} \quad i \notin I(l). \end{cases} \tag{6.6}$$

Hence,

$$@D = \bigcup_{i \in I} @D_i, \tag{6.7}$$

and therefore (see, e.g., [10, Lemma 2.1])

$$\operatorname{dist}(x^0, @D) = \min_{i \in I} \operatorname{dist}(x^0, @D_i). \tag{6.8}$$

Observe now that, by (6.2),

$$\operatorname{dist}(x^0, @D_i) = \operatorname{dist}(x^0, U_i) \qquad (i \in I), \tag{6.9}$$

where

$$U_i := \operatorname{bd}_0 D_i = \begin{cases} \{x \in \mathbb{R}^I_+ \mid l_i x_i = 1\} & \text{if } i \in I(l) \\ \varnothing & \text{if } i \notin I(l). \end{cases} \tag{6.10}$$

Hence, by (6.8)–(6.10) $\operatorname{dist}(x^0, \varnothing) = +\infty$ and Lemma 6.1, we obtain (6.4). ▮

COROLLARY 6.1. *Let G be a closed normal set, $x^0 \in @G$, $g^0 := \min P_G(x^0) \neq 0$, and*

$$D^0 := \left\{ x \in \mathbb{R}^I_+ \,\middle|\, \left\langle \frac{1}{g^0}, x \right\rangle \geqslant 1 \right\} = \left\{ x \in \mathbb{R}^I_+ \mid \min_{i \in I(g^0)} \frac{x_i}{g_i^0} \geqslant 1 \right\}. \tag{6.11}$$

Then

$$\operatorname{dist}(x^0, \operatorname{bd}_0 D^0) = \operatorname{dist}(x^0, G). \tag{6.12}$$

Proof. Observe first that $x^0 \in \operatorname{int}_0 D^0$; i.e.,

$$\left\langle \frac{1}{g^0}, w^0 \right\rangle = \min_{i \in I(g^0)} \frac{x_i^0}{g_i^0} > 1. \tag{6.13}$$

Indeed, by $g^0 \neq 0$, (3.6) and (3.1) we have $I_+ = I(g^0) \neq \varnothing$ and $x_i^0 - g_i^0 = r > 0$, whence $x_i^0 > g_i^0$, for all $i \in I_+ = I(g^0)$. Hence, by Theorem 6.1, we obtain

$$\operatorname{dist}(x^0, \operatorname{bd}_0 D^0) = \min_{i \in I_+} (x_i^0 - g_i^0), \tag{6.14}$$

whence, by (3.1), the result follows. ▮

Remark 6.1. (a) We have

$$g^0 = \min P_{\mathrm{bd}_0 D^0}(x^0). \tag{6.15}$$

Indeed, by $\langle 1/g^0, g^0 \rangle = 1$, we have $g^0 \in \mathrm{bd}_0 D^0$. Also, by (6.12), $\mathrm{dist}(x^0, \mathrm{bd}_0 D^0) = \mathrm{dist}(x^0, G) = \|x^0 - g^0\|$, and hence

$$g^0 \in P_{\mathrm{bd}_0 D^0}(x^0). \tag{6.16}$$

But, by the definition (6.11) of D^0, for each $x \in D^0$ (and hence, in particular, for each $x \in P_{\mathrm{bd}_0 D^0}(x^0)$) we have $g^0 \leqslant x$, which, together with (6.16), yields (6.17).

(b) Actually, the last observation of (a) is a particular case of the following fact: *For any* $l \in \mathbb{R}^I_+ \setminus \{0\}$ *and* $D := \{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \geqslant 1\}$ *there holds*

$$\frac{1}{l} = \min D. \tag{6.17}$$

Indeed, by $\langle l, \frac{1}{l} \rangle = 1$, we have $\frac{1}{l} \in D$. Also, by the definition of D, for each $x \in D$ we have $x_i \geqslant (\frac{1}{l})_i$ for all $i \in I(l)$, whence also for all $i \in I$, so $x \geqslant \frac{1}{l}$.

COROLLARY 6.2. *Let G be a closed normal set, $x^0 \in @G$, and $g^0 := \min P_G(x^0)$. If g^0 is a weak Pareto point of G, then*

$$\mathrm{dist}(x^0, G) = \max_{H \in \mathscr{H}_1} \mathrm{dist}(x^0, H) = \max_{H \in \mathscr{H}_2} \mathrm{dist}(x^0, H), \tag{6.18}$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ denote the families of all lower min-type closed half-spaces which separate G and $B_0(x^0, r)$, with $r := \|x^0 - g^0\|$, respectively, which contain G and do not contain x^0.

Proof. Let $H \in \mathscr{H}_1$. Then, since $G \subseteq H$, we have $\mathrm{dist}(x^0, G) \geqslant \mathrm{dist}(x^0, H)$, which proves the inequality $\geqslant$ in the first part of (6.18). On the other hand, since g^0 is a w.P. point of G, by Theorem 4.1 the set $H^0 := \mathrm{bd}_0 D^0 \cup @D^0$, with D^0 of (6.11), is a lower min-type closed half-space separating G and $B_0(x^0, r)$; i.e., $H^0 \in \mathscr{H}_1$. Also, by Theorem 6.1, we have $\mathrm{dist}(x^0, G) = \mathrm{dist}(x^0, H^0)$. The proof of the second equality in (6.18) is similar. ∎

COROLLARY 6.3. *Let G be a normal set and let $x^0 \in @G$. If g^0 is a weak Pareto point of G, then*

$$\operatorname{dist}(x^0, G) = \max_{\substack{l \in R^I_+ \\ \langle l, g\rangle \leqslant 1 < \langle l, x^0\rangle \ (g \in G)}} \min_{i \in I(l)} \left(x_i^0 - \frac{1}{l_i} \right). \tag{6.19}$$

Proof. This follows from Corollary 6.2 and Theorem 6.1, since $H := \{x \in \mathbb{R}^I_+ \mid \langle l, x\rangle \leqslant 1\} \in \mathscr{H}_2$ if and only if $\langle l, g\rangle \leqslant 1 < \langle l, x^0\rangle (g \in G)$. ∎

7. BEST APPROXIMATION BY CONORMAL SETS

7.1. *Some Tools from Abstract Convex Analysis*

Let us recall some concepts and results from abstract convex analysis (see, e.g., [8]), which we shall need in the rest of the paper.

Let X and W be two (non-empty) sets. A mapping $\Delta: 2^X \to 2^W$ is called a *duality* (where 2^X denotes the family of all subsets of X) if

$$\Delta(G) = \bigcap_{g \in G} \Delta(\{g\}) \qquad (G \subseteq X). \tag{7.1}$$

The *dual* of Δ is the duality $\Delta': 2^W \to 2^X$ defined by

$$\Delta'(P) = \{x \in X \mid P \subseteq \Delta(\{x\})\} \qquad (P \subseteq W). \tag{7.2}$$

A subset G of X is said to be $\Delta'\Delta$-*convex*, in symbols, $G \in \mathscr{C}(\Delta'\Delta)$, if $\Delta'\Delta(G) = G$, or, what is equivalent, if for each $x \in @G := X \backslash G$ there exists $w \in W$ such that

$$G \subseteq \Delta'(\{w\}), \qquad x \in X \backslash \Delta'(\{w\}). \tag{7.3}$$

For any duality $\Delta: 2^X \to 2^W$ and any function $f: X \to \overline{\mathbb{R}} = [-\infty, +\infty]$, *the conjugate of type Lau* (called also the "level set conjugate") *of f associated to* Δ is the function $f^{\mathscr{L}(\Delta)}: W \to \overline{\mathbb{R}}$ defined by

$$f^{\mathscr{L}(\Delta)}(w) = - \inf_{x \in X \backslash \Delta'(\{w\})} f(w) \qquad (w \in W). \tag{7.4}$$

If $f: X \to \overline{\mathbb{R}}$ and $x^0 \in X$ are such that $f(x^0) \in \mathbb{R}$ and $\Delta: 2^X \to 2^W$ is a duality, the *subdifferential of f at* x^0 *with respect to* Δ is the subset $\partial^{\mathscr{L}(\Delta)} f(x^0)$ of W defined by

$$\partial^{\mathscr{L}(\Delta)}f(x^0)=\{w^0\in W \mid x^0\in X\backslash\Delta'(\{w^0\}),\, f(x^0)=-f^{\mathscr{L}(\Delta)}(w^0)\}$$
$$=\{w^0\in W \mid x^0\in X\backslash\Delta'(\{w^0\}),\, f(x^0)=\min_{x\in X\backslash\Delta'(\{w^0\})} f(x)\}. \tag{7.5}$$

Observe that, by Proposition 2.3, the closed normal sets are the elements of $\mathscr{C}(\Delta_1'\Delta_1)$, where $X=W=\mathbb{R}_+^I$ and $\Delta_1: 2^X\to 2^W$ is the duality defined by

$$\Delta_1(G):=\{l\in\mathbb{R}_+^I \mid \langle l, g\rangle\leqslant 1\ (g\in G)\} \qquad (G\subseteq R_+^I). \tag{7.6}$$

Also, by Proposition 2.2, the normal sets are the elements of $\mathscr{C}(\Delta_2'\Delta_2)$, where $X=W=R_+^I$ and $\Delta_2: 2^X\to 2^W$ is the duality defined by

$$\Delta_2(G):=\{l\in\mathbb{R}_+^I \mid \langle l, g\rangle< 1\ (g\in G)\} \qquad (G\subseteq R_+^I). \tag{7.7}$$

In the next section we shall use the following two results (where $@G:=X\backslash G$):

THEOREM A [11, Corollary 6; 10, Theorem 5.2]. *Let X and W be two sets, $\Delta: 2^X\to 2^W$ a duality, $f: X\to\overline{\mathbb{R}}$, and $G\in\mathscr{C}(\Delta'\Delta)$. Then*

$$\inf f(@G)=\inf_{w\in\Delta(G)}\inf f(X\backslash\Delta'(\{w\})). \tag{7.8}$$

THEOREM B [10, Theorem 6.2]. *Let X and W be two sets, $\Delta: 2^X\to 2^W$ a duality, $f: X\to\overline{\mathbb{R}}$, and $G\in\mathscr{C}(\Delta'\Delta)$. For an element $x^0\in @G$ with $f(x^0)\in\mathbb{R}$, the following statements are equivalent*:

1°. *We have*

$$f(x^0)=\inf f(@G). \tag{7.9}$$

2°. *There exists $w^0\in\Delta(G)\cap\partial^{\mathscr{L}(\Delta)}f(x^0)$ such that*

$$f^{\mathscr{L}(\Delta)}(w^0)=\max f^{\mathscr{L}(\Delta)}(\Delta(G)). \tag{7.10}$$

7.2. *The Distance to an Upper Min-Type Half-Space, and a Conormal Set*

Let $l=(l_1, ..., l_n)\in\mathbb{R}_+^I$, and let us consider the upper min-type half-space D of (6.1). Furthermore, let

$$x^0=(x_1^0, ..., x_n^0)\in @D=\{x\in\mathbb{R}_+^I \mid \min_{i\in I(l)} l_ix_i<1\}. \tag{7.11}$$

The following result, corresponding to Theorem 6.1, gives an explicit formula for $\operatorname{dist}(x^0, D)$.

THEOREM 7.1. *For D and x^0 as in* (6.1) *and* (7.11), *we have*

$$\operatorname{dist}(x^0, D) = \max_{i\in I(l)}\left(\frac{1}{l_i} - x_i^0\right) = \max_{\substack{i\in I(l)\\ (1/l_i)-x_i^0>0}}\left(\frac{1}{l_i} - x_i^0\right). \tag{7.12}$$

Proof. The last equality in (7.12) is obvious by (7.11).
By (6.1), we have (6.5), with D_i of (6.6). Hence, since $D\subseteq D_i$ $(i\in I)$,

$$\operatorname{dist}(x^0, D) \geqslant \max_{i\in I} \operatorname{dist}(x^0, D_i). \tag{7.13}$$

Let us observe now that if $x^0\notin D_i$, then, by (6.6), we have $i\in I(l)$, and hence, by Lemma 6.1,

$$\operatorname{dist}(x^0, D_i) = \begin{cases} \dfrac{1}{l_i} - x_i^0 & \text{if } x^0\notin D_i \\ 0 & \text{if } x^0\in D_i. \end{cases} \tag{7.14}$$

Consequently, by (7.13), (7.14), and (7.11), we obtain

$$\operatorname{dist}(x^0, D) \geqslant \max_{\substack{i\in I\\ x^0\notin D_i}} \operatorname{dist}(x^0, D_i) = \max_{\substack{i\in I(l)\\ (1/l_i)-x_i^0>0}}\left(\frac{1}{l_i} - x_i^0\right). \tag{7.15}$$

In order to prove the opposite inequality, define $y^0\in\mathbb{R}_+^I$ by

$$y_i^0 := \begin{cases} \dfrac{1}{l_i} & \text{if } x^0\notin D_i \\ x_i^0 & \text{if } x^0\in D_i. \end{cases} \tag{7.16}$$

We claim that $y^0\in D$, i.e., $\min_{i\in I(l)} l_i y_i^0 \geqslant 1$. Indeed, if $i\in I(l)$ and $x^0\notin D_i$, then, by (7.16), we have $l_i y_i^0 = 1$, while if $i\in I(l)$ and $x^0\in D_i$, then, by (7.16) and (6.6), we have $l_i y_i^0 = l_i x_i^0 \geqslant 1$. This proves the claim $y^0\in D$. Hence, by $y^0\in D$, (7.16), and (7.11), we obtain

$$\operatorname{dist}(x^0, D) \leqslant \|x^0 - y^0\| = \max\Big(\max_{\substack{i\in I\\ x^0\notin D_i}} |x_i^0 - y_i^0|, 0\Big)$$

$$= \max_{\substack{i\in I(l)\\ (1/l_i)-x_i^0>0}}\left(\frac{1}{l_i} - x_i^0\right),$$

which, together with (7.15), yields (7.12). ∎

Now let $G \subseteq \mathbb{R}^I_+$ be a normal set, and let $\tilde{x}^0 \in G$, $\operatorname{dist}(\tilde{x}^0, @G) > 0$. In order to give an explicit formula for $\operatorname{dist}(\tilde{x}^0, @G)$, we shall apply Theorem A of Section 7.1 to $X = W = \mathbb{R}^I_+$, the duality $\Delta_2 : 2^{\mathbb{R}^I_+} \to 2^{\mathbb{R}^I_+}$ of (7.7), and the function $f_{\tilde{x}^0} : \mathbb{R}^I_+ \to \mathbb{R}$ defined by

$$f_{\tilde{x}^0}(x) := \|\tilde{x}^0 - x\| \qquad (x \in \mathbb{R}^I_+). \tag{7.17}$$

THEOREM 7.2. *Let* $G \subseteq \mathbb{R}^I_+$ *be a normal set, and let* $\tilde{x}^0 \in G$, $\operatorname{dist}(\tilde{x}^0, @G) > 0$. *Then*

$$\operatorname{dist}(\tilde{x}^0, @G) = \inf_{\substack{l \in \mathbb{R}^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} \max_{i \in I(l)} \left(\frac{1}{l_i} - \tilde{x}^0_i \right). \tag{7.18}$$

Proof. Since G is a normal set, by Proposition 2.2 we have $G \in \mathscr{C}(\Delta_2' \Delta_2)$, with Δ_2 of (7.7). Hence, by Theorem A of Section 7.1, applied to $f = f_{\tilde{x}^0}$ of (7.17), (7.7), (7.2), and Theorem 7.1, we have

$$\begin{aligned}
\operatorname{dist}(\tilde{x}^0, @G) &= \inf_{l \in \Delta_2(G)} \operatorname{dist}(\tilde{x}^0, \mathbb{R}^I_+ \setminus \Delta_2'(\{l\})) \\
&= \inf_{\substack{l \in \mathbb{R}^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} \operatorname{dist}(\tilde{x}^0, \{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \geqslant 1\}) \\
&= \inf_{\substack{l \in \mathbb{R}^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} \max_{i \in I(l)} \left(\frac{1}{l_i} - \tilde{x}^0_i \right). \quad \blacksquare
\end{aligned}$$

7.3. *Nearest Points in Conormal Sets*

We have the following result corresponding to Theorem 3.1:

THEOREM 7.3. *Let* $G \subseteq \mathbb{R}^I_+$ *be a normal set, and let* $x^0 \in G$, $r := \operatorname{dist}(x^0, @G) > 0$. *If* $P_{@G}(x^0) \neq \emptyset$ (*this happens, e.g., when* G *is open*), *then* $y^0 := \max P_{@G}(x^0)$ (*i.e., the greatest element* y^0 *of* $P_{@G}(x^0)$) *exists, namely*

$$y^0 = x^0 + r\mathbf{1}. \tag{7.19}$$

Proof. Let $y \in P_{@G}(x^0)$. Then $\|x^0 - y\| = r$, whence $y \leqslant x^0 + r\mathbf{1}$. Hence, since $@G$ is a conormal set, $x^0 + r\mathbf{1} \in @G$. Thus, since $\|x^0 - (x^0 + r\mathbf{1})\| = r$, we have $x^0 + r\mathbf{1} \in P_{@G}(x^0)$. Consequently, since we have seen above that $y \leqslant x^0 + r\mathbf{1}$ for all $y \in P_{@G}(x^0)$, it follows that $x^0 + r\mathbf{1} = \max P_{@G}(x^0)$. $\blacksquare$

Let us given now a characterization of nearest points in $@G$.

THEOREM 7.4. *Let $G \subseteq \mathbb{R}^I_+$ be a normal set, and let $\tilde{x}^0 \in G$. For an element $x^0 \in @G$ the following statements are equivalent*:

1°. *There holds*

$$\|\tilde{x}^0 - x^0\| = \operatorname{dist}(\tilde{x}^0, @G). \tag{7.20}$$

2°. *There exists $l^0 \in R^I_+$ with $\langle l^0, x^0 \rangle \geqslant 1$, such that*

$$\langle l^0, g \rangle < 1 \qquad (g \in G), \tag{7.21}$$

$$\max_{i \in I(l)} \left(\frac{1}{l^0_i} - \tilde{x}^0_i \right) = \min_{\substack{l \in R^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} \max_{i \in I(l)} \left(\frac{1}{l_i} - \tilde{x}^0_i \right). \tag{7.22}$$

Proof. Since G is a normal set, by Proposition 2.2 we have $G \in \mathscr{C}(\Delta'_2 \Delta_2)$, with Δ_2 of (7.7). Hence, by Theorem B of Section 7.1, we have 1° if and only if there exists $l^0 \in \Delta_2(G) \cap \partial^{\mathscr{L}(\Delta_2)} f(x^0)$ such that

$$f^{\mathscr{L}(\Delta_2)}(l^0) = \max f^{\mathscr{L}(\Delta_2)}(\Delta_2(G)), \tag{7.23}$$

where $f = f_{\tilde{x}^0}$ of (7.17); i.e., by (7.5) and (7.4), there exists an element $l^0 \in R^I_+$ with $\langle l^0, x^0 \rangle \geqslant 1$ satisfying (7.21) and

$$-\operatorname{dist}(\tilde{x}^0, \mathbb{R}^I_+ \setminus \Delta'_2(\{l^0\})) = \max_{\substack{l \in \mathbb{R}^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} (-\operatorname{dist}(\tilde{x}^0, \mathbb{R}^I_+ \setminus \Delta'_2(\{l\}))). \tag{7.24}$$

But, by (7.7) and Theorem 7.1, for any $l \in \mathbb{R}^I_+$ we have

$$\operatorname{dist}(\tilde{x}^0, \mathbb{R}^I_+ \setminus \Delta'_2(\{l\})) = \operatorname{dist}(\tilde{x}^0, \{x \in \mathbb{R}^I_+ \mid \langle l, x \rangle \geqslant 1\}) = \max_{i \in I(l)} \left(\frac{1}{l_i} - \tilde{x}^0_i \right),$$

and thus (7.24) means that

$$-\max_{i \in I(l)} \left(\frac{1}{l^0_i} - \tilde{x}^0_i \right) = \max_{\substack{l \in \mathbb{R}^I_+ \\ \langle l, g \rangle < 1\ (g \in G)}} \left(-\max_{i \in I(l)} \left(\frac{1}{l_i} - \tilde{x}^0_i \right) \right),$$

which is nothing else than (7.22). ∎

Using Theorem 7.4 above, we obtain

COROLLARY 7.1. *If G is a normal set, $\tilde{x}^0 \in G$, and* $\operatorname{dist}(\tilde{x}^0, @G)$ *is attained for some x^0, then the* inf *on the right hand side of* (7.18) *is attained for some l^0.*

Note added in proof. One can show that (4.15) implies (4.14). Hence, in Theorem 4.3 condition (4.14) can be omitted.

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